Problem Set 4, Solutions

This problem set is due Friday, 13 February, at 4 PM. Please **staple** your problem set together and deliver it to John.

Although much of your solutions will be mathematical in nature, your write-up should be as much text as equations/mathematics (perhaps more). Correct answers, poorly justified, will not be worth many points. Things you should explain include: where you found a given equation, which law (eg, "Newton's Second Law") you are using, what mathematical steps you are following, and what assumptions you are making. Also note that your solutions will be easier to follow (and less prone to errors) if you work in analytic equations for as long as possible before you plug in any numbers given in the problem. (This has the added benefit of being generalized so you can plug in new numbers easily if you need to.)

A note on collaboration/help/references: please cite any source (this includes the text, although you can just say, '...Equation 10 in Taylor...') you use as well as acknowledging any help you may have received from the tutors or John. I encourage you to work with buddies, but you must acknowledge their assistance. Furthermore, your write-up must be your own and you must understand everything in it. Failure to acknowledge or cite a source of help is a form of academic dishonesty and will be dealt with accordingly. Also note that looking up solutions (either using previous years' solutions, a previous year's student's solutions, or finding textbook solutions) is strictly forbidden. These problems were designed to aid you in your education. To avoid them is doing yourself a disservice.

Finally, an acknowledgment: many of the problems in this problem set are due in whole or in part to the great Bill Titus of Carleton. Any mistakes are mine, but the good stuff is his.

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0. Feedback (**4 pts**)
   Estimate how long this problem set took you. Also include any questions or comments you have for John about this problem set.
1. Balloon Boy (**23 pts**)
   a) (**5 pts**)
      The mass of a 6-year-old is around 22 kg according to the charts at the CDC website (`www.cdc.gov/growthcharts/clinical_charts.htm`). Student numbers may vary, although a significant variation from this probably bears scrutiny.
      As for the tarp, we'll be generous and assume that it's basically two discs, each radius 3.05 m, sewn together at the seams. (This is almost certainly an underestimate since the inflated result wouldn't be 6.1 m in diameter, but we'll run with it.) To get the mass, we take the area, πr^2, times the surface mass density, given as 0.25 kg/m^2, times 2 (for the two parts). So the mass is about 15 kg.
      Finally, the plywood. I found resources that say that a 4'x8' piece of 1/4" plywood is 25 lb. One square-foot is therefore 1/32 of that, or 0.78 lb. However, I specified 1/2" plywood, so we'll double this to get about 1.5 lb. Converting to metric, a square meter (about 11 times the area of a square foot) has a mass of about 7 kg.
      Adding this all up, we get a total mass to be lifted of around 44 kg. Under the acceleration of standard Earth gravity (using Newton's second law, weight $= mg$), this is about 430 N of weight.
   b) (**4 pts**)
      The volume of an ellipsoid is given by $V = \frac{4}{3}\pi abc$, where a, b, and c are the radii. I have the diameters, so the radii are $a = b = 3.05$ m and $c = 0.75$ m. Thus, we have a volume of $V = 29$ m^3.

Helium has a density of 0.1786 kg/m³ at STP, air has a density of 1.2 kg/m³, so the buoyancy is $gV(\rho_{\text{air}} - \rho_{\text{He}}) = (9.8 \text{ m/s}^2)(29 \text{ m}^3)(1 \text{ kg/m}^3) = 286N$.

c) (**4 pts**) A mile in altitude is 1.6 km. So using the formula given

$$\begin{aligned} \rho_{\text{air}}(z) &= \rho_{\text{air}}(0)e^{-z/H} \\ &= (1.2 \text{ kg/m}^3)e^{-z/(8.5 \text{ km})} \end{aligned} \quad (1)$$

So for $z = 1.6$ km, $\rho_{\text{air}} = 1$ kg/m³. The difference in the densities of helium and air is therefore 0.8 kg/m³, so the balloon can lift $gV\Delta\rho = (9.8 \text{ m/s}^2)(29 \text{ m}^3)(0.8 \text{ kg/m}^3) = 227$ N.

d) (**10 pts**)
In general, it seems improbable that the boy was ever in the balloon. A good paragraph will explain (briefly, but clearly) how they calculated the lifting capacity and what they got. A comparison to the estimated weight of the boy+balloon should then be made.

2. Concept Questions (**14 pts**)
For each of the following questions, just give your answer. No explanation is required. Questions are based on Eric Mazur's *Peer Instruction: A User's Guide*.

a) (ii) the same
b) (iii) It went down
c) (iii) The fluid levels are still the same
d) (ii) smaller than
e) (ii) B
f) (iv) They're all the same.
g) (iii) 4

3. Particle in a Potential (**21 pts**)

a) (**3 pts**)
Since $F(x) = -dV(x)/dx$,

$$\begin{aligned} F(x) &= -\frac{d}{dx}\left(-\frac{cx}{x^2+a^2}\right) \\ &= \frac{c}{a^2+x^2} - \frac{2cx^2}{(a^2+x^2)^2} \\ &= \frac{c(x^2-a^2)}{(a^2+x^2)^2} \end{aligned} \quad (2)$$

b) (**5 pts**)
I'll use $d_c = a$, so that $u = x/d_c = x/a$. Plugging that in,

$$\begin{aligned} V(ua) &= -\frac{cua}{(ua)^2+a^2} \\ &= -\frac{c}{a}\frac{u}{u^2+1} \end{aligned} \quad (3)$$

Now we can make a dimensionless potential energy, $U(u) = aV(ua)/c$, since c/a seems to be the energy scale. (This was somewhat apparent in the original expression for $V(x)$ where we needed

c/a to have energy units, but it's a lot more obvious now.

$$U(u) = V(ua)/c = -\frac{u}{1+u^2} \quad (4)$$

We can graph that easily.

c) (**3 pts**)

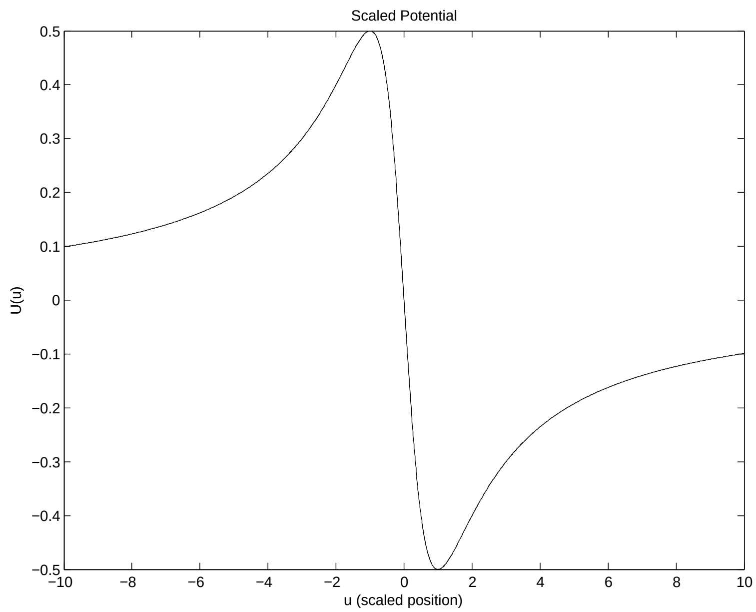

Figure 1: Potential of particle, scaled to unitless variables. (Potential is scaled by $E_c = c/a$ and position is scaled by $d_c = a$.)

d) (**10 pts**)
(2 pts each)

- If the total energy is $E = -c/2a$, the particle is at the stable equilibrium at the bottom. Here there is no motion and the particle will happily stay put.
- If $-c/2a < E < 0$, the particle will be trapped around the equilibrium point. The distance it's allowed to go in either direction is set by how large E actually is; a larger E allows more range, especially on the right where it tends toward infinity as $E \to 0$. The particle will oscillate back and forth around the equilibrium, moving fastest at the equilibrium point and slowing as it approaches the turn-around points.
- If $0 < E < c/2a$, the particle can move freely from either $-\infty$ to a stopping point on the left side of the peak or from the right side of the peak to $+\infty$. In the former case, the particle will move fastest at $-\infty$ and slow to a stop as it moves up the peak. If the particle is moving

right initially, it will turn around and go left after encountering the peak. If it is moving left to start with, it will keep going that way forever since there is nothing to make it stop and reverse direction.
If the particle is to the right of the peak, the behavior is basically the same, with directions reversed. The particle will go particularly fast as it passes through the stable equilibrium point, of course.

- If $E = c/2a$, the particle could sit at the unstable equilibrium at the peak on the left. However, any slight disturbance will dislodged it and make it move away from this point. If the particle is coming toward the peak from the left or the right, it will reach this unstable equilibrium point, but only in infinite time.
- If $E > c/2a$, the particle can move freely through the system. It will continue in the same direction as it starts in, speed and slowing as it moves through the valley and the peak.

Note that the particle cannot have energy less than $-c/2a$.

4. Euler's Identity (**14 pts**)
Solutions here are just math. Few words, other than descriptions of steps, are required.

a) **(2 pts)**
For this, we'll need to recall that $|x + iy| = \sqrt{x^2 + y^2}$.

$$\begin{aligned} \left|e^{i\theta}\right| &= |\cos(\theta) + i\sin(\theta)| && \text{Euler's identity} \\ &= \sqrt{(\cos(\theta))^2 + (\sin(\theta))^2} && \text{Definition modulus, given above} \\ &= 1 && \text{Because } \cos^2 + \sin^2 = 1 \end{aligned} \tag{5}$$

So we've proven what we set out to show.

b) **(2 pts)**
Recall that the complex conjugate is given by $(x + iy)^* = x - iy$.
Then

$$\begin{aligned} \left(e^{i\theta}\right)^* &= (\cos(\theta) + i\sin(\theta))^* && \text{Euler's Identity} \\ &= \cos(\theta) - i\sin(\theta) && \text{Definition of conjugate} \\ &= \cos(-\theta) + i\sin(-\theta) && \text{Sine is odd and cosine is even, so this is OK} \\ &= e^{-i\theta} && \text{Euler's identity} \end{aligned} \tag{6}$$

c) **(4 pts)**
We'll need our angle summation formulas, here: $\cos(a + b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ and $\sin(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b)$.

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) && \text{Euler's Identity} \\ &= \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2) && \text{Expanding the binomials} \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) && \text{Angle summation formulas} \\ &= e^{(\theta_1+\theta_2)i} && \text{Euler's identity} \end{aligned} \tag{7}$$

d) (**3 pts**)

$$\begin{aligned}
\frac{de^{i\theta}}{d\theta} &= \frac{d}{d\theta}\big(\cos(\theta) + i\sin(\theta)\big) && \text{Euler's Identity} \\
&= -sin(\theta) + i\cos(\theta) && \text{Taking the derivatives} \\
&= i(\cos(\theta) + i\sin(\theta)) && \text{Factoring an } i \text{ out} \\
&= ie^{i\theta} && \text{Euler's Identity}
\end{aligned} \tag{8}$$

e) (**3 pts**)
We could technically use the fundemental theorem of calculus and the above result to do this, but we're asked to do it with Euler's Identity. So:

$$\begin{aligned}
\int e^{i\theta} d\theta &= \int \cos(\theta) + i\sin(\theta) d\theta && \text{Euler's Indentity} \\
&= \int \cos(\theta) d\theta + \int i\sin(\theta) d\theta && \text{Splitting the integral} \\
&= \int \cos(\theta) d\theta + i\int \sin(\theta) d\theta && \text{Factoring a constant } i \text{ outside} \\
&= \sin(\theta) - i\cos(\theta) + C && \text{Integrating} \\
&= -i(\cos(\theta) + i\sin(\theta)) + C && \text{Factoring out } -i \\
&= -ie^{i\theta} + C && \text{Euler's Identity}
\end{aligned} \tag{9}$$

f) (**3 pts**)
Let's just do a replacement and see what happens

$$\begin{aligned}
\sinh(i\theta) &= \frac{e^{i\theta)} - e^{-i\theta}}{2} && \text{Definition} \\
&= \frac{1}{2}\left((\cos(\theta) + i\sin(\theta)) - (\cos(-\theta) + i\sin(-\theta))\right) && \text{Euler's identity} \\
&= \frac{1}{2}\left(\cos(\theta) + i\sin(\theta) - \cos(\theta) + i\sin(\theta))\right) && \text{Using even/oddness of cosine/sine} \\
&= \frac{1}{2}\left(2\sin(\theta)\right) && \text{Combining/canceling} \\
&= i\sin(\theta) && \text{We're done}
\end{aligned} \tag{10}$$

g) (**3 pts**) Let's again do a replacement and see what happens

$$\begin{aligned}
\cosh(i\theta) &= \frac{e^{i\theta)} + e^{-i\theta}}{2} && \text{Definition} \\
&= \frac{1}{2}\left((\cos(\theta) + i\sin(\theta)) + (\cos(-\theta) + i\sin(-\theta))\right) && \text{Euler's identity} \\
&= \frac{1}{2}\left(\cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta))\right) && \text{Using even/oddness of cosine/sine} \\
&= \frac{1}{2}\left(2\cos(\theta)\right) && \text{Combining/canceling} \\
&= \cos(\theta) && \text{We're done}
\end{aligned} \tag{11}$$